

The Quantum Hall Fluid and Non-Commutative Chern Simons Theory

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Abstract

The first part of this paper is a review of the author's work with S. Bahcall which gave an elementary derivation of the Chern Simons description of the Quantum Hall effect for filling fraction $1/n$. The notation has been modernized to conform with standard gauge theory conventions.

In the second part arguments are given to support the claim that abelian non-commutative Chern Simons theory at level n is exactly equivalent to the Laughlin theory at filling fraction $1/n$. The theory may also be formulated as a matrix theory similar to that describing D0-branes in string theory. Finally it can also be thought of as the quantum theory of mappings between two non-commutative spaces, the first being the target space and the second being the base space.

1 Fluid Dynamics in Co-moving Coordinates

The configuration space of charged particles in a strong magnetic field ¹ is a non-commutative space. The charged particles behave unlike conventional relativistic or non-relativistic particles in so much as they are locked in place by the large magnetic field. By contrast, a neutral system such as a dipole [1, 2] does move like a conventional particle although it grows in size with its momentum. Such dipoles are the objects described by non-commutative field theory.

If we are considering a system of charged particles in a strong magnetic field we may either describe the system in terms of charge carrying fields such as the electron field or in terms of neutral fields such as the density and current. The former fields correspond to the frozen particles but the latter fields carry the quantum numbers of dipoles. This suggests that the currents and density of a system of electrons in a strong magnetic field may be described by a non-commutative quantum field theory. In this paper we will show that this is indeed the case and that the Laughlin electron theory of the fractional quantum hall states is equivalent to a non-commutative Chern Simons theory describing the density and current.

The first six sections of the paper review work done in 1991 with Safi Bahcall [3]. The purpose is to give an elementary description of the fractional Quantum Hall fluid for the case in which the inverse filling fraction is an integer, and to explain how in these cases, the long distance behavior of the Quantum Hall fluid can be described by Chern Simons theory [4].

Both the odd and even integer cases describe quantum hall states, the odd cases corresponding to fermions and the even to bosons.

In the remaining sections it is shown that precise quantitative agreement with Laughlin's theory [5] can be obtained if the ordinary Chern Simons theory is replaced by the non-commutative theory [6]. Alternatively it can be formulated as a matrix theory similar to that describing D0-branes in string theory [8].

We will begin with a description of a dissipationless fluid. Consider a collection of identical non-relativistic particles, indexed by α , moving on a plane with Lagrangian

$$L = \sum_{\alpha} \frac{m}{2} \dot{x}_{\alpha}^2 - U(x) \tag{1.1}$$

¹A magnetic field is considered strong if the energy scales of interest are too low for higher Landau levels to be excited or admixed into the wave function.

where U is the potential energy. Assuming the system behaves like a fluid we can pass to a continuum description by replacing the discrete label α by a pair of continuous coordinates y_1, y_2 . These coordinates label the material points of the fluid and move with it ². They are the analog of co-moving coordinates in cosmology. The system of particles is thereby replaced by a pair of continuum fields $x_i(y, t)$ with $i = 1, 2$. Without loss of generality we can choose the coordinates y so that the number of particles per unit area in y space is constant and given by ρ_0 . The real space density is

$$\rho = \rho_0 \left| \frac{\partial y}{\partial x} \right| \quad (1.2)$$

where $\left| \frac{\partial y}{\partial x} \right|$ is the Jacobian connecting the x and y coordinate systems.

The potential U is assumed to arise out of short range forces which lead to an equilibrium when the real space density is ρ_0 . Thus in equilibrium the Jacobian is 1. With these assumptions and conventions the Lagrangian can be written as

$$L = \int d^2y \rho_0 \left[\frac{m}{2} \dot{x}^2 - V \left(\rho_0 \left| \frac{\partial y}{\partial x} \right| \right) \right] \quad (1.3)$$

where the potential energy has now been expressed in terms of the density ³.

The Lagrangian (1.3) has an exact gauge invariance under area preserving diffeomorphisms of the y plane. Consider any area preserving diffeomorphism from y to y' with unit Jacobian. The fluid field x transforms as a scalar, $x'(y') = x(y)$. It is easily seen that (1.3) is invariant. To find the consequences of this invariance consider an infinitesimal transformation

$$y'_i = y_i + f_i(y). \quad (1.4)$$

The x 's transform as

$$\delta x_a = \frac{\partial x_a}{\partial y_i} f_i(y). \quad (1.5)$$

The condition for f to represent an infinitesimal area preserving diffeomorphism is that

$$f_i = \epsilon_{ij} \frac{\partial \Lambda(y)}{\partial y_j} \quad (1.6)$$

²Fluid mechanics described in the y coordinates is called the Lagrangian description. The Eulerian description expresses the fluid properties as functions of x .

³We are considering the case of zero temperature. More generally the potential can also be a function of the temperature. However, in order to study sound waves with small amplitudes we also should impose the "adiabatic condition". This will lead to a potential which is ρ dependent. I am grateful to M.M. Sheikh Jabbari for pointing this out.

with Λ being an arbitrary gauge function. Equation (1.5) then takes the form

$$\delta x_a = \epsilon_{ij} \frac{\partial x_a}{\partial y_i} \frac{\partial \Lambda}{\partial y_j} \quad (1.7)$$

2 Kelvin's Circulation Theorem and Vortices

Since the transformation (1.7) is a symmetry of L , a conserved quantity exists and is given by

$$\int d^2y \Pi_a \delta x_a \quad (2.1)$$

where Π_a is the canonical conjugate to x_a , proportional to \dot{x}_a . Thus for any Λ

$$\int d^2y \rho_0 \left[\epsilon_{ij} \dot{x}_a \frac{\partial x_a}{\partial y_i} \frac{\partial \Lambda}{\partial y_j} \right] \quad (2.2)$$

is conserved. Integrating by parts gives

$$\frac{d}{dt} \int d^2y \epsilon_{ij} \frac{\partial}{\partial y_j} \left[\dot{x}_a \frac{\partial x_a}{\partial y_i} \right] \Lambda = 0. \quad (2.3)$$

Since eq(2.3) is true for all Λ we can conclude that

$$\frac{d}{dt} \left[\frac{\partial}{\partial y_j} \left(\epsilon_{ij} \dot{x}_a \frac{\partial x_a}{\partial y_i} \right) \right] = 0 \quad (2.4)$$

To see how Kelvin's circulation theorem comes about, integrate (2.4) over an area bounded by a closed curve Γ . Using Gauss's theorem we find

$$\frac{d}{dt} \oint_{\Gamma} \dot{x}_a dx_a = 0. \quad (2.5)$$

Thus conservation of circulation follows from gauge invariance in the same way that Gauss' law is derived in free electrodynamics.

In free electrodynamics points where $\nabla \cdot E$ is not zero correspond to static charges. In a similar way points where $\frac{\partial}{\partial y_j} \left(\epsilon_{ij} \dot{x}_a \frac{\partial x_a}{\partial y_i} \right) \neq 0$ are vortices which are frozen into the fluid. The vortex-free fluid satisfies

$$\left[\frac{\partial}{\partial y_j} \left(\epsilon_{ij} \dot{x}_a \frac{\partial x_a}{\partial y_i} \right) \right] = 0 \quad (2.6)$$

3 Electromagnetic Analog

The analogy with electromagnetic theory can be made much closer by restricting attention to small motions of the fluid. Assuming the potential V in (1.3) has a minimum at $\rho = \rho_0$, there is a time independent solution of the equations of motion given by

$$x_i = y_i. \quad (3.1)$$

Now consider small deviations from this equilibrium solution parameterized by a vector field A defined by

$$x_i = y_i + \epsilon_{ij} \frac{A_j}{2\pi\rho_0} \quad (3.2)$$

Working to linear order the gauge transformation (1.7) becomes

$$\delta A_i = 2\pi\rho_0 \frac{\partial \Lambda}{\partial y_i} \quad (3.3)$$

which has the standard form of an abelian gauge transformation. The exact form of the transformation is

$$\delta A_i = 2\pi\rho_0 \frac{\partial \Lambda}{\partial y_i} + \frac{\partial A_i}{\partial y_l} \frac{\partial \Lambda}{\partial y_m} \epsilon_{l,m}. \quad (3.4)$$

The second nonlinear term in (3.4) is suggestive of a non-commutative structure for the field theory describing the Quantum Hall fluid. We will return to this in section (7).

We assume that for small deviations of the density from its equilibrium value the potential has the form

$$V = \mu \left(\rho_0 \left| \frac{\partial y}{\partial x} \right| - \rho_0 \right)^2. \quad (3.5)$$

The density to lowest order is

$$\rho = \rho_0 - \frac{1}{2\pi} (\nabla \times A) \quad (3.6)$$

and the Lagrangian takes the form

$$L = \frac{1}{g^2} \int d^2y \frac{1}{2} \left[\dot{A}^2 - \frac{2\mu\rho_0^2}{m} (\nabla \times A)^2 \right]. \quad (3.7)$$

where the coupling constant g is defined by

$$g^2 = (2\pi)^2 \frac{\rho_0}{m} \quad (3.8)$$

The Lagrangian (3.7) is the familiar Maxwell Lagrangian in temporal gauge. The velocity of light is given by

$$c^2 = \frac{2\mu\rho_0^2}{m}. \quad (3.9)$$

The photons of the analog electrodynamics are just sound waves in the fluid.

The vortex free condition (2.6) in the linearized approximation becomes the Gauss law constraint

$$\nabla \cdot E = \nabla \cdot \dot{A} = 0. \quad (3.10)$$

Thus we see that charges in the gauge theory represent vortices.

The equations of motion and the Gauss law constraint can be derived from a single action principle by introducing a time component for the vector field A . The procedure is well known and will not be repeated here.

The derivation of the gauge description of fluid mechanics given in this section was done in the temporal gauge $A_0 = 0$. However once we have introduced A_0 back into the equations we are free to work in other gauges. To some degree this freedom allows us to relax the condition that the motion of the fluid be a small deviation from the configuration $x = y$. In fact we may use the gauge freedom to work in a gauge in which the displacement of the fluid is as small as possible. To carry this out let us introduce a positive measure M , for the magnitude of A .

$$M = \int d^2y A(y) \cdot A(y). \quad (3.11)$$

We may choose our gauge by requiring that M be as small as possible, that is

$$\delta M = 0 \quad (3.12)$$

where the variation is with respect to an arbitrary gauge transformation $\delta A = \nabla \lambda$. Thus for a given configuration A should be chosen to satisfy

$$\int d^2y A \cdot \nabla \lambda = 0. \quad (3.13)$$

This obviously requires the Coulomb gauge.

$$\nabla \cdot A = 0 \quad (3.14)$$

Thus by working in the Coulomb gauge we are also insuring that the y and x coordinates agree as closely as possible. More generally when the non-linearity and non-commutativity of the equations is included we will define a generalization of the Coulomb gauge which minimizes

$$M = \int d^2y (x - y)^2. \quad (3.15)$$

4 Charged Fluid in A Magnetic Field

Now let us assume that the particles making up the fluid are electrically charged and move in a background magnetic field B . For a point particle of charge e in a uniform magnetic field the Lagrangian gets an extra term

$$\frac{eB}{2}\epsilon_{ab}\dot{x}_a x_b \quad (4.1)$$

For a fluid with charge to mass ratio e/m the extra term is

$$L' = \frac{eB}{2} \int \rho_0 d^2y \epsilon_{ab} \dot{x}_a x_b. \quad (4.2)$$

Note that the canonical momentum density conjugate to x_a is given by

$$\Pi_a = \frac{\partial L'}{\partial \dot{x}_a} = \frac{eB\rho_0}{2}\epsilon_{ab}x_b \quad (4.3)$$

Substituting (3.2) into (4.2) and dropping total time derivatives gives

$$L' = \frac{eB}{8\pi^2\rho_0} \int d^2y \epsilon_{ab} \dot{A}_a A_b. \quad (4.4)$$

This has the usual form of an abelian Chern Simons Lagrangian in the temporal gauge. Among its effects are to give the photon a mass. The mass is given by

$$m_{photon} = eB/m \quad (4.5)$$

which will be recognized as the cyclotron frequency.

In the absence of a magnetic field, the role of static charges was to represent the fluid vortices. When the B field is turned on the long range behavior of the theory is dominated by the Chern Simons term and the character of the charges changes. In what follows we will be mainly interested in the long distance behavior of the Quantum Hall effect. In this case we may drop the Maxwell term completely. Let us do so.

The Lagrangian (4.2) is invariant under area preserving diffeomorphisms. Accordingly (2.1) is still conserved, but now the canonical momentum conjugate to x_a is $\Pi_a \propto \epsilon_{ab}x_b$. The conserved gauge generator is

$$\frac{1}{2} \frac{\partial}{\partial y_j} \left\{ \epsilon_{ij} \epsilon_{ab} x_b \frac{\partial x_a}{\partial y_i} \right\} = \frac{1}{2} \epsilon_{ij} \epsilon_{ab} \frac{\partial x_b}{\partial y_j} \frac{\partial x_a}{\partial y_i}$$

This is just the Jacobian from x to y which is given by ρ_0/ρ . It therefore follows that the density of the fluid at a fixed co-moving point y is time independent.

In the absence of vortices (quasiparticles in the Quantum Hall context) the conserved generator is set to unity. Thus the equations of motion are supplemented with the constraint

$$\frac{1}{2}\epsilon_{ij}\epsilon_{ab}\frac{\partial x_b}{\partial y_j}\frac{\partial x_a}{\partial y_i}=1 \quad (4.6)$$

The equations of motion and constraint can be obtained from a single action by introducing a time component of A and replacing the ordinary time derivative in (4.2) by an appropriate covariant derivative:

$$L' = \frac{eB\rho_0}{2}\epsilon_{ab}\int d^2y \left[\left(\dot{x}_a - \frac{1}{2\pi\rho_0}\{x_a, A_0\} \right) x_b + \frac{\epsilon_{ab}}{2\pi\rho_0}A_0 \right]. \quad (4.7)$$

In this equation we have introduced the Poisson bracket notation

$$\{F(y), G(y)\} = \epsilon_{ij}\partial_i F \partial_j G$$

Now return to the linearized approximation for small oscillations of the fluid. Using the expression (3.6) for the density we see that the conservation law requires the “magnetic field”⁴ at each point y , to be time independent. The analog of a vortex is a δ function magnetic field:

$$\nabla \times A = 2\pi\rho_0 q \delta^2(y) \quad (4.8)$$

where q measures the strength of the vortex⁵. The solution to this equation is unique up to a gauge transformation. In the Coulomb gauge, $\nabla \cdot A = 0$, it is given by

$$A_i = q\rho_0\epsilon_{ij}\frac{y_j}{y^2}. \quad (4.9)$$

Since $\epsilon_{ij}A_j/2\pi\rho_0$ is the displacement of the fluid we see that the Chern Simons vortex is really a radial displacement of the fluid toward or away from the vortex-center by an amount $q/2\pi r$ depending on the sign of q . This implies either an excess or deficit of ordinary electric charge at the vortex. The magnitude of this excess/deficit is

$$e_{qp} = \rho_0 q e. \quad (4.10)$$

The charged vortex is the Laughlin quasiparticle[5].

⁴Warning: Do not confuse the analog magnetic field $\nabla \times A$ with the external magnetic field B

⁵Again we warn the reader not to confuse quantities in the analog gauge theory with real electromagnetic quantities. The real electric charge of an electron is e and the analog gauge charge of the vortex is q . We will see that the quasiparticle also carries a real electric charge

To further understand the quasiparticle we must quantize the fluid. We will not carry out a full quantization but instead rely on elementary semiclassical methods. Assume the fluid is composed of particles of charge e . If Π_a is the momentum density then

$$p_a = \frac{\Pi_a}{\rho_0} = eB\epsilon_{ab}x_b/2 \quad (4.11)$$

is the momentum of a single particle. The standard Bohr–Sommerfeld quantization condition is

$$\oint p_a dx_a = 2\pi n. \quad (4.12)$$

The quantization condition (4.12) becomes

$$eB \oint \frac{\epsilon_{ab}x_b}{2} dx_a = 2\pi n. \quad (4.13)$$

The integral in (4.13) is the real (x – *space*) area of the region. To interpret the meaning of this equation we shall assume that any change in the properties of the fluid within the closed curve, such as the introduction of a quasiparticle can only change $eB \times (\text{area})$ by 2π times an integer. Thus

$$\frac{eB}{2\pi\rho_0} \oint A_a dy_a = 2\pi n. \quad (4.14)$$

Using the vortex solution (4.9) then gives

$$eBq = 2\pi n \quad (4.15)$$

From (4.10) a single elementary quasiparticle ($n = 1$) has electric charge

$$e_{qp} = 2\pi \frac{\rho_0}{B} \quad (4.16)$$

This agrees with the value of the quasiparticle charge from Laughlin’s theory [5].

According to (4.9) the vector potential diverges at the vortex. To correctly understand the physics very close to the origin we must give up the approximation of small disturbances. Doing so we will see that the solution is well behaved. The correct equation for the vortex is obtained by modifying (4.6) to include a source,

$$\frac{1}{2}\epsilon_{ij}\epsilon_{ab}\frac{\partial x_b}{\partial y_j}\frac{\partial x_a}{\partial y_i} - 1 = q\delta^2(y). \quad (4.17)$$

This equation has the solution

$$x_i = y_i \sqrt{1 + \frac{q}{\pi|y|^2}}. \quad (4.18)$$

Far from the origin the solution agrees with (4.8) but has a more interesting behavior near $y = 0$. Although the vortex is a point in y space it has finite area in x space. The leading behavior is given by

$$x_i \sim \sqrt{\frac{q}{\pi}} \frac{y_i}{|y|}. \quad (4.19)$$

The point $y = 0$ is mapped to a circle of radius $\sqrt{q/\pi}$, leaving an empty hole in the center. The hole has area q and an electric charge deficit $\rho_0 q e/m$.

Before continuing we will introduce some notation which will be helpful in relating the parameters eB and ρ_0 with field theoretic parameters. The quantity

$$\nu \equiv \frac{2\pi\rho_0}{eB} \quad (4.20)$$

is the ratio of the number of electrons to the magnetic flux and is called the filling fraction. Comparing (4.4) with the conventional Chern Simons notation we see that $1/\nu$ is the Chern Simons *level* usually called k . In terms of the filling fraction, the quasiparticle charge (4.16) is

$$e_{qp} = e\nu \quad (4.21)$$

The parameter

$$\theta \equiv \frac{1}{2\pi\rho_0} \quad (4.22)$$

appearing in (3.2) will later be identified with the non-commutativity parameter of the non-commutative coordinates that replace the y 's in section (6).

5 Fractional Statistics of Quasiparticles

In this section we will give an elementary explanation of why quasiparticles have fractional statistics [5]. The fractional statistics question can be reduced to the calculation of the Berry phase induced by transporting one quasiparticle around another. The calculation of this phase depends only on the fact that when a quasiparticle is created in the fluid it pushes the fluid out by a distance $q/2\pi r$ where r is the distance from the quasiparticle.

In the undisturbed fluid a quasiparticle at the origin can be created by a unitary operator having the form

$$U(0) = \exp \left\{ \frac{iq}{2\pi} \int d^2y \frac{\Pi(r) \cdot r}{r^2} \right\} \quad (5.1)$$

where Π is the canonical conjugate to x and r is the distance from the origin. A similar operator $U(a)$ can be constructed which creates a quasiparticle at $y = a$. It is important to remember that the operator $U(0)$ not only creates a quasiparticle at $y = 0$ but also pushes the fluid away by distance $q/2\pi|r|$.

Now let us construct a pair of quasiparticles one at point a and one at b . The naive guess would be

$$|a, b\rangle = U(a)U(b)|0\rangle. \quad (5.2)$$

However this is not right. The first operator to act, $U(b)$, creates a quasiparticle at $x = y = b$. Then $U(a)$ acts to create a quasiparticle at $x = y = a$ but it also pushes the fluid so that the first quasiparticle ends up at a shifted position. The right way to create the quasiparticles is to compensate for this effect by shifting the argument of the first operator;

$$|a, b\rangle = U(a)U(b - d_{a,b})|0\rangle. \quad (5.3)$$

where

$$d_{a,b} = \frac{q}{2\pi} \frac{a - b}{|a - b|^2}. \quad (5.4)$$

This time the creation of the quasiparticle at a pushes the center of the first quasiparticle to its correct location at $x = b$.

Let us consider the Berry phase picked up by the wave function when the quasiparticle at b is transported around a circle centered at the fixed point a .

$$\Gamma_{a,b} = \oint \left\langle a, b \left| \frac{\partial}{\partial b} \right| a, b \right\rangle \quad (5.5)$$

which can also be written

$$\Gamma_{a,b} = \oint (\langle a, b | a, b + db \rangle - 1). \quad (5.6)$$

The inner product $\langle a, b | a, b + db \rangle$ is given by

$$\begin{aligned} \langle 0 | U^\dagger(b - d_{b,a}) U^\dagger(a) U(a) U(b + db - d_{a,b+db}) | 0 \rangle = \\ \langle 0 | U^\dagger(b - d_{b,a}) U(b + db - d_{a,b+db}) | 0 \rangle. \end{aligned} \quad (5.7)$$

This last expression, when inserted into (5.6) gives the phase for a state with only one quasiparticle moved in a circle of radius smaller by $|d_{ab}|$. Writing $R = |a - b|$ and $\Delta R = d_{a,b}$ we have

$$\Gamma_{a,b}(\text{loop with radius } R) = \Gamma_b(\text{loop with radius } R - \Delta R). \quad (5.8)$$

Thus the extra phase due to the presence of the quasiparticle at a is

$$\Delta\Gamma = \Gamma_b(R) - \Gamma_b(R - \Delta R). \quad (5.9)$$

To compute $\Gamma_b(R)$ is easy because it is just the phase due to moving a charge in a uniform magnetic field. It is the product of the charge times the enclosed flux. Since the charge of the quasiparticle is $Q = q\rho_0 e$ the difference in phase is

$$\Delta\Gamma = 2\pi R \Delta R Q B = \rho_0 e B q^2. \quad (5.10)$$

Inserting the quantized value of q from (4.14), $q = 2\pi/eB$ and defining the filling fraction ν

$$\nu = \frac{2\pi}{eB} \rho_0 \quad (5.11)$$

we find

$$\Delta\Gamma = 2\pi\nu. \quad (5.12)$$

The parameter ν is the ratio of the particle number to the magnetic flux and will be recognized as the usual filling fraction. The connection between filling fraction and Berry phase given in (5.12) is the same as derived by Laughlin and is equivalent to the usual anyon statistics for the quasiparticle.

6 Quantization of the Filling Fraction

The simplest fractional Quantum Hall states are those for which the filling fraction ν is of the form $1/n$ with n being an integer. Furthermore if the charged particles comprising the fluid are fermions (bosons) then the integer n should be odd (even). This is a quantum mechanical effect related to angular momentum quantization. We can very roughly see how it comes about by considering a pair of nearest neighbor particles in the fluid. The relative angular momentum of the pair is

$$L_{1,2} = \frac{1}{2} \epsilon_{ab} (x_1 - x_2)_a (p_1 - p_2)_b \quad (6.1)$$

Using $p_a = \frac{eB}{2} \epsilon_{ab} x_b$ we find

$$L_{1,2} = \frac{1}{4} e B \delta^2. \quad (6.2)$$

where δ is the separation between neighbors. In the ground state it satisfies

$$\delta \sim \sqrt{1/\rho_0} \quad (6.3)$$

and

$$L_{1,2} \sim eB/\rho_0. \quad (6.4)$$

If we require this to be an odd (fermions) or even (bosons) integer we get

$$eB/\rho_0 \sim n \quad (6.5)$$

or

$$\nu \sim 1/n. \quad (6.6)$$

It should be clear that this argument is at best a heuristic suggestion of why we may expect a quantization of the inverse filling fraction. To find the precise quantization condition requires a more exact quantization of the fluid such as that provided by the Laughlin wave functions. In the next section we will show that exact agreement with the Laughlin theory can be obtained by a generalization of the fluid model in which ordinary abelian Chern Simons theory is replaced by Chern Simons theory on a non-commutative space [6]. The main result is that we will rigorously find the connection between filling fraction and particle statistics demanded by Laughlin's theory.

7 Non-Commutative Chern Simons Theory

Let us consider the full nonlinear fluid equations which result from the Lagrangian (4.7), the gauge invariance (3.4) and the constraint (4.6). We introduce the Poisson bracket notation

$$\{F(y), G(y)\} = \epsilon_{ij} \partial_i F \partial_j G \quad (7.1)$$

The Lagrangian then takes the form [7]

$$L' = \frac{1}{4\pi\nu} \epsilon_{\mu\nu\rho} \left[\frac{\partial A_\mu}{\partial y_\rho} - \frac{\theta}{3} \{A_\mu, A_\rho\} \right] A_\nu \quad (7.2)$$

where the indices (μ, ν, ρ) run over $(0, 1, 2)$ and $\theta = 1/2\pi\rho_0$.

Similarly the gauge invariance and the constraint take the form

$$\delta A_a = \frac{\partial \lambda}{\partial y_a} + \theta \{A_a, \lambda\} \quad (7.3)$$

and

$$\epsilon_{ab} \left[\partial_a A_b - \frac{\theta}{2} \{A_a, A_b\} \right] = 0 \quad (7.4)$$

The nonlinear theory defined by (7.2), (7.3) and (7.4) is a Chern Simons gauge theory based on the group of area preserving diffeomorphisms (APD's) of the parameter space y_i . In the real electron system the y space is just a convenient and approximate way to label the electrons. Area preserving transformations are merely a way of re-labeling the particles. The correct way to label electrons is with a discrete index and the re-labeling symmetry is the permutation group. The gauge theory of APD's captures many of the long distance features of the Quantum Hall system but it does not capture the discrete or granular character of the electron system.

There is a well known way of discretizing the APD's. Two equivalent lines of reasoning lead to the same conclusion. The first is to recognize that (7.2), (7.3) and (7.4) are first order truncations of a non-commutative theory, that is a field theory on a non-commutative y -space. This theory is the non-commutative version of Chern Simons theory with spatial non-commutativity. It is defined by the Lagrangian ⁶

$$L_{NC} = \frac{1}{4\pi\nu} \epsilon_{\mu\nu\rho} \left(\hat{A}_\mu * \partial_\nu \hat{A}_\rho + \frac{2i}{3} \hat{A}_\mu * \hat{A}_\nu * \hat{A}_\rho \right) \quad (7.5)$$

with the usual Moyal star-product defined in terms of a non-commutativity parameter

$$\theta = \frac{1}{2\pi\rho_0}. \quad (7.6)$$

The Lagrangian (7.2) is identical to (7.5) expanded to first order in θ . On the other hand the full non-commutative theory is defined on a non-commutative space in which there is a discrete indivisible unit of y -space area which can be identified with the electron.

A point worth discussing is which of the two sets of coordinates x or y are non-commutative. The answer is both but in different senses. The non-commutative Chern Simons theory that we are considering is defined by fields that are functions of the y coordinates, that is the y space is the base space. The Moyal brackets are defined in terms of y derivatives. Evidently it is the y coordinates which are non-commutative with non-commutativity parameter θ . The basic quantum of y -area is θ and it represents the area occupied by a single electron. The non-commutativity of the y space is classical and not due to the quantization of the field theory.

On the other hand the Lagrangian (4.2) makes plain the fact that the x space is also non-commutative but in the quantum sense. From (4.11) we see

⁶We have used the notation of Seiberg and Witten in which non-commutative gauge fields are "hatted"

that the momentum conjugate to the coordinates are proportional to the coordinates themselves. Indeed the coordinates x_1, x_2 do not commute as quantum objects. In this case the non-commutativity parameter is $1/eB$ which is proportional to the area occupied, not by an electron, but by a single quantum of magnetic flux. Evidently the non-commutative Chern Simons theory describes mappings between two non-commutative spaces.

Another route to the same theory which emphasizes the discrete particle aspects of the fluid begins with a matrix theory representation of the electrons ⁷ in a manner similar to the construction of the matrix theory of D0-branes [8]. We replace the classical configuration space of K electrons by a space of two $K \times K$ hermitian matrices X_a . The time component of the vector potential is also replaced by an hermitian matrix. We will eventually let K be infinite. The natural action, generalizing (4.7) to matrix theory in a background magnetic field is [9, 10, 11]

$$L' = \frac{eB}{2} \epsilon_{ab} \text{Tr} \left(\dot{x}_a - i[x_a, \hat{A}_0]_m \right) \dot{x}_b + eB\theta \hat{A}_0. \quad (7.7)$$

In this equation the notation $[f, g]_m$ indicates f and g are classical matrices and the subscript m means that the commutator is evaluated in the classical matrix space and not in the Hilbert space of quantum mechanics. Quantum commutators will be denoted in the usual way with no subscript.

The equation of constraint is obtained by varying this action with respect to \hat{A}_0 . We find

$$[x_a, x_b]_m = i\theta \epsilon_{ab} \quad (7.8)$$

It is well known that (7.8) can only be solved with infinite matrices. Therefore we must allow the number of electrons K to be infinite.

Now choose two definite matrices y_a satisfying

$$[y_a, y_b]_m = i\theta_{ab} = i\theta \epsilon_{ab} \quad (7.9)$$

For example such matrices can be easily constructed from harmonic oscillator creation and annihilation operators. We can also represent (7.9) in the form

$$y_2 = -i\theta \frac{\partial}{\partial y_1}$$

⁷The use of matrix theory in this paper is different from that in [9, 14]. In that case the electrons were described by string ends and the matrix theory described the units of magnetic flux or D0-branes. In this paper the electrons are described by matrix theory as if they were D0-branes. The relation between the two descriptions will be discussed in a forthcoming paper with N. Toumbas and B. Freivogel.

Next define the matrices \hat{A}_a

$$x_a = y_a + \epsilon_{ab}\theta\hat{A}_b. \quad (7.10)$$

Inserting (7.10) into (7.7) gives the non-commutative Chern Simons Lagrangian (7.5). Thus, from two points of view we see that non-commutative Chern Simons theory is connected with the physics of charges moving in a magnetic field.

8 Statistics of the Chern Simons Particles

If non-commutative Chern Simons theory describes particles in a magnetic field, what kind of particles are they? In particular are they fermions, bosons, anyons or something new? The answer as we will see depends on the level of the Chern Simons theory $1/\nu$.

The particles described by Matrix Theory [8] satisfy a more general statistics than either Fermi or Bose statistics. The permutation of particle labels is replaced by the bigger group of unitary transformations in the space of the matrix indices. However certain backgrounds may break the unitary symmetry to the subgroup of permutations. In that case the transformation property under the subgroup will determine the statistics. In this section we will compute the statistics of the particles defined by the matrix theory of the previous section.

Let us begin with identification of canonical variables from the action (7.7). The canonical momentum conjugate to matrix entry $(x_1)_{mn}$ is

$$(p_1)_{nm} = \frac{eB}{2}(x_2)_{nm}. \quad (8.1)$$

The *quantum* commutation relations are not between x_1 and x_2 but between their matrix entries

$$[(x_1)_{mn}, (x_2)_{rs}] = \frac{2i}{eB}\delta_{ms}\delta_{nr}. \quad (8.2)$$

Another way to express this is

$$(x_2)_{mn} = \frac{-2i}{eB} \frac{\partial}{\partial(x_1)_{nm}}. \quad (8.3)$$

Thus unlike the y 's which are classical non-commuting coordinates, the matrix components of the x 's are non-commutative in the quantum sense.

In order to simplify our notation we will work in the Hilbert space basis in which $(x_1)_{mn}$ is diagonal. We will call $(x_1)_{mn}$ and $(x_2)_{mn}$ X_{mn} and $\frac{-2i}{eB} \frac{\partial}{\partial X_{nm}}$ or more simply $(x_2)_{mn} = (2eB^{-1})P_{nm}$.

We can now rewrite the constraint equation (7.8) in the form

$$X_{mn}P_{nr} - P_{mn}X_{nr} = ieB\theta\delta_{mr} = \frac{i}{\nu}\delta_{mr} \quad (8.4)$$

Since this equation was derived by varying with respect to A_0 it should be interpreted like the Gauss law constraint as acting on a wave function whose arguments are X_{mn} .

$$\{X_{mn}P_{nr} - P_{mn}X_{nr}\}|\Psi\rangle = \frac{i}{\nu}\delta_{mr}|\Psi\rangle \quad (8.5)$$

This is the fundamental set of equations determining the ground state of the non-commutative Chern Simons theory.

The left hand side of (8.5) has a familiar form. It resembles an angular momentum operator, that is a generator of rotations. In fact it is the quantum mechanical generator that generates unitary transformations among the matrix entries. For example consider an Hermitian matrix λ that generates infinitesimal transformations according to

$$\delta X = i[X, \lambda]_m. \quad (8.6)$$

The corresponding quantum generator is

$$\Lambda = \lambda_{rm} \{X_{mn}P_{nr} - P_{mn}X_{nr}\} \quad (8.7)$$

and the constraint (8.5) becomes

$$\Lambda|\Psi\rangle = \frac{1}{\nu}Tr\lambda|\Psi\rangle \quad (8.8)$$

Now let us turn to the question of the statistics of the charged particles comprising the quantum hall fluid. Let us consider the operation of exchanging two particles. Consider the case of just two matrix theory particles described by 2×2 matrices. The unitary matrix describing their interchange is obviously

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (8.9)$$

More generally for $K \times K$ matrices the exchange of the m^{th} and n^{th} particle can be written as a matrix with two nonzero elements $U_{nm} = U_{mn} = 1$, and all other elements equal to zero. Furthermore we can also write

$$U = \exp i\lambda \quad (8.10)$$

such that $Tr\lambda = \pi$. Equation (8.8) takes the form

$$\Lambda|\Psi\rangle = \frac{\pi}{\nu}|\Psi\rangle \quad (8.11)$$

Now consider the exchange operation on the Hilbert space of states. Call the unitary quantum operator which implements the exchange operation on the space of states \mathcal{P}_{mn} .

$$\mathcal{P}_{mn} = \exp i\Lambda. \quad (8.12)$$

Evidently

$$\begin{aligned} \mathcal{P}_{mn}|\Psi\rangle &= \exp i\Lambda|\Psi\rangle \\ &= \exp\left(\frac{i\pi}{\nu}\right)|\Psi\rangle. \end{aligned} \quad (8.13)$$

This is a remarkable formula. It says that if the filling fraction satisfies $\nu = 1/(2n+1)$ the charged particles are fermions while if $\nu = 1/(2n)$ they are bosons! This of course is identical to the statistics–density connection implied by Laughlin’s wave function.

For more general values of ν the simple Laughlin wave functions

$$\Psi_{laugh} = \prod (Z_i - Z_j)^{\frac{1}{\nu}} \exp\left(-\frac{1}{2} \sum Z^* Z\right) \quad (8.14)$$

should not be interpreted as quantum hall states for either fundamental fermions or bosons. Their correct interpretation is as wave functions for a system of anyons in a magnetic field. Thus we expect that for non–integer $1/\nu$ the non-commutative cs theory has an interpretation in terms of Quantum Hall states for fundamental charged anyons. It is an interesting question how to describe the non–integer filling of electrons by non-commutative field theory.

It is important to keep in mind that quantities defined in the y frame of reference are not gauge invariant. Consider as an example the real space particle density defined in (3.6) and its non–linear generalizations. This quantity is the density measured in x coordinates but it is naturally viewed as a function of y in the formal field theory. However functions of y are not observables, indeed they are not gauge invariant since y itself changes under the area preserving diffeomorphisms. The gauge invariant quantity is the density at a point in x space. Thus

$$\rho(x) = \rho(y + \epsilon\theta A) \quad (8.15)$$

is gauge invariant while $\rho(y)$ is not. The density is closely related to the non-commutative field strength. In the fluid equations of section (3) the density is given by

$$\rho = \rho_0(1 + \theta F)^{-1} \quad (8.16)$$

where

$$F = \epsilon_{ab} \left(\frac{\partial A_b}{\partial y_a} + \frac{\theta}{2} \{A_a, A_b\} \right).$$

In the more exact non-commutative theory the field strength \mathcal{F} is given by

$$\hat{F} = \epsilon_{ab} \left(\frac{\partial \hat{A}_b}{\partial y_a} + \frac{1}{2} (\hat{A}_a * \hat{A}_b - \hat{A}_b * \hat{A}_a) \right) \quad (8.17)$$

Since the physical density is given by (8.16) we should expect that the value of the field strength at a location x is a gauge invariant physical quantity. By contrast its value at a definite value of y is not. In fact it is well known that local quantities in a non-commutative gauge theory are not gauge invariant.

Obviously the gauge invariant quantity can be obtained by a Taylor series expansion in the θ parameter. Let us denote the value of the field strength at the point x by $f(x)$.

$$f(x) = \hat{F}(y_i + \epsilon_{ij}\theta\hat{A}_j) = \hat{F}(y) + \epsilon_{ij}\theta\hat{A}_j\partial_i\hat{F} + \dots \quad (8.18)$$

This expansion is closely related to the Seiberg Witten [12, 13] map ⁸ which relates the gauge dependent non-commutative field strength to a gauge invariant commutative field strength. The Seiberg Witten map has the form

$$F(x) = \hat{F} + \theta\hat{F}^2\hat{F}(y) + \epsilon_{ij}\theta\hat{A}_j\partial_i\hat{F} + \dots \quad (8.19)$$

Where in this equation F is a gauge invariant commutative field strength. Evidently to the order we are working

$$f(x) = F - \theta F^2. \quad (8.20)$$

The physical gauge invariant correlation functions are of the form

$$\langle 0|f(x)f(x')|0\rangle \quad (8.21)$$

Calculating these correlation functions is obviously non-trivial. The correlators of $\hat{F}(y)$ are more straightforward. Carrying out the Seiberg Witten map is difficult except as a power series in θ . This may be worth doing in order to compare with Laughlin's theory which relates these correlation functions

to density-density correlations in a well defined Coulomb gas in a neutralizing background. One interesting prediction is that as the filling fraction decreases a phase transition occurs in which the fractional Quantum Hall states give way to a Wigner Crystal. This implies that the non-commutative Chern Simons theory also has a transition to a new phase at large level.

⁸This connection was explained to me by Nick Toumbas.

9 The Phase Transition

If there is a new phase at low filling fraction it is likely to be associated with the breaking of a symmetry that we have not yet discussed. To understand it let us return to the single particle Lagrangian (4.1). This Lagrangian has a symmetry under area preserving diffeomorphisms (APD's) of x space. We emphasize that the group of APD's of x space is an entirely different symmetry than the gauge symmetry of area preserving transformations of y space.

To see the symmetry under x -space APD's consider the infinitesimal APD

$$\begin{aligned} x'_i &= x_i + \delta_i \\ \delta_i &= \epsilon_{ij} \frac{\partial S(x)}{\partial x_j} \end{aligned} \tag{9.1}$$

where S is a function of x . An easy calculation reveals that the variation of the Lagrangian is a total time derivative:

$$\delta L = \frac{d}{dt} \left[x_m \frac{\partial S}{\partial x_m} - 2S \right]. \tag{9.2}$$

Hence the theory is invariant under APD of x space. This fact is unchanged when we pass to the fluid Lagrangian (4.2). The theory therefore exhibits APD invariance of two kinds; one which acts on the base coordinates y , and the other which acts on the target coordinates x .

When we pass to the quantum theory of a particle in a magnetic field the APD are replaced by the corresponding transformations on a non-commutative x space. That the x space is non-commutative is clear from the fact that the two components x_1, x_2 are canonical conjugates of one another. The quantum transformations which replace the classical APD's are the unitary transformations on the Hilbert space of LLL's.

As an example consider the quadratic functions

$$S = c_{ij} x_i x_j \tag{9.3}$$

with c_{ij} being traceless and symmetric. In this case the total time derivative vanishes and the Lagrangian is invariant. The APD in this case are linear transformations. For the case $S = x_1 x_2$ the finite transformations have the form

$$\begin{aligned} x'_1 &= c x_1 \\ x'_2 &= \frac{1}{c} x_2. \end{aligned} \tag{9.4}$$

That is they squeeze one direction and stretch the orthogonal direction.

We can construct the corresponding symmetries in the non-commutative quantum theory most easily by focusing on the matrix version of the theory. Consider the generator

$$S = \text{Tr} c_{ij} x_i x_j. \quad (9.5)$$

Using the quantum commutation relations (8.2) we find the matrix valued equation

$$\delta x_l = i[x_l, S] = \frac{2}{eB} \epsilon_{jl} c_{ij} x_i \quad (9.6)$$

This is the matrix version of the APD generated by S .

It is straightforward to show that S commutes with the equations of constraint (8.4). Thus the Unitary transformation generated by S is a symmetry of the theory. It is therefore important to know how the symmetry is realized. We will argue that the incompressible Quantum Hall fluid is invariant under transformations such as (9.4).

One reason for believing this is that the Quantum Hall fluid has a uniform density which does not change under any APD. There is no obvious contradiction with saying the fluid is invariant.

Quantum mechanically the transformations (9.4) induce a unitary transformation in the space of LLL's. For example the single particle wave function

$$\psi = e^{-\frac{1}{2}Z^\dagger Z}$$

gets mapped to

$$\psi' = e^{-\frac{1}{2}Z^\dagger Z} e^{\frac{1}{2}\alpha Z^2}$$

where α is a parameter representing the amount of squeezing and stretching. By translating ψ around on the plane we can construct an (over) complete set of LLL's both before squeezing or after. It is not difficult to prove that the average electron occupation number for electrons in the original states ψ and the squeezed states ψ' are the same.

For the case $\nu = 1$ it is easy to see that the Quantum Hall state is invariant under unitary transformations of the LLL's. This is because the property having all fermion states filled is basis independent. A fully filled system of fermion levels is fully filled in any basis. We will assume without further proof that the all Laughlin states are invariant under the unitary transformations of LLL's, at least for the squeezing/stretching transformations.

As was mentioned, when the filling factor becomes sufficiently small the system makes a transition to another phase, the Wigner crystal phase. The transition is not driven by

energetics. Indeed the transition can be seen in the behavior of the Laughlin wave functions themselves. While it may be correct that the relevance of the Laughlin functions depends on the existence of repulsive forces, the precise form of the wave functions corresponds to vanishing potential.

In any case a crystal-like phase can not be invariant under general APD's. It seems likely then that the crystal phase is associated with a spontaneous breaking of the x-space APD's. It is obvious that an APD acting on a crystal will change it and take us to a new configuration. This in turn implies that the solutions to the constraint equations will have to become degenerate. It would be good to see this directly from the equations of the non-commutative Chern Simons theory but at the moment it is a conjecture.

Another point may be very relevant in this context. The existence of the transition must be related to an instability of the homogeneous fluid phase. Recent work has shown a generic tendency for non-commutative quantum field theories to exhibit transitions to striped and other inhomogeneous phases. [15].

10 Non-Commutative Quasiparticles

Let us consider the non-commutative generalization of (4.17) which defined a quasiparticle. The generalization of the left hand side of (4.17) is obvious.

$$\epsilon_{ij} \frac{\partial x_b}{\partial y_j} \frac{\partial x_a}{\partial y_i} \rightarrow \theta[x_b, x_a]_m.$$

The more interesting question is how to represent the delta function $\delta^2(y)$ on the non-commutative space. The correct answer is that a delta function should be replaced by a projection operator onto a particular vector in the matrix space [16]. In order to carry this out in detail let us introduce a particular description of the infinite dimensional matrices. We begin by labeling basis vectors ⁹ $|m\rangle$ where m runs over the positive integers and zero. We also introduce matrices a and a^\dagger with the usual properties

$$\begin{aligned} a^\dagger |n\rangle &= \sqrt{n+1} |n+1\rangle \\ a |n\rangle &= \sqrt{n} |n-1\rangle \end{aligned} \tag{10.1}$$

Equation (7.9) is satisfied by expressing the y 's in terms of Fock space matrices;

$$y_1 = \sqrt{\frac{\theta}{2}}(a + a^\dagger)$$

⁹The notation $|n\rangle$ will be used for vectors in the matrix space. For vectors in the quantum space of states we use $|\Psi\rangle$.

$$y_2 = \sqrt{\frac{\theta}{2}}i(a - a^\dagger). \quad (10.2)$$

A delta function at the origin may be represented as a projection operator onto the vector $|0\rangle$

$$\theta\delta(y) \rightarrow |0\rangle\langle 0|. \quad (10.3)$$

The constraint equation becomes

$$\theta^{-1}[x_1, x_2]_m = i + i\nu |0\rangle\langle 0|. \quad (10.4)$$

This equation is equivalent to the classical non-commutative field equation for the quasi-particle. The analysis of this equation has similarities with that in [16] where classical soliton solutions of non-commutative field theory were found.

It is not difficult to solve (10.3) exactly. First note that if we drop the quasiparticle term the constraint is solved by

$$\begin{aligned} x_1 &= y_1 = \sqrt{\frac{\theta}{2}}(a + a^\dagger) \\ x_2 &= y_2 = \sqrt{\frac{\theta}{2}}i(a - a^\dagger). \end{aligned} \quad (10.5)$$

We can introduce two new matrices b, b^\dagger defined by the action

$$\begin{aligned} b^\dagger |n\rangle &= \sqrt{n+1+\nu} |n+1\rangle \\ b |n\rangle &= \sqrt{n+\nu} |n-1\rangle \quad \text{for } n \neq 0 \\ b |0\rangle &= 0 \end{aligned} \quad (10.6)$$

If we now set

$$\begin{aligned} x_1 &= \sqrt{\frac{\theta}{2}}(b + b^\dagger) \\ x_2 &= \sqrt{\frac{\theta}{2}}i(b - b^\dagger). \end{aligned} \quad (10.7)$$

we find that (10.4) is satisfied.

It is also possible to see that the solution is given in the generalized Coulomb gauge as defined by minimizing the matrix version of (3.15).

$$\begin{aligned} \delta M &= 0 \\ M &= \text{Tr}(x_i - y_i)^2. \end{aligned} \quad (10.8)$$

In this equation the a variation of x is defined by

$$\delta x = i[\lambda, x]_m \quad (10.9)$$

where λ is an hermitian matrix. The variational condition leads to the gauge condition

$$\sum_{i=1,2} [x_i, y_i]_m = 0. \quad (10.10)$$

From (10.2) and (10.7) the condition may be written as

$$[b, a^\dagger]_m + [b^\dagger, a]_m = 0. \quad (10.11)$$

This can easily be confirmed from the defining properties of the operators. Thus we have found an exact Coulomb gauge classical solution of abelian non-commutative Chern Simons theory at level 1. Multiple quasiparticle solutions are easy to find but we will not do so here.

11 Conclusions

In this paper we have reviewed the derivation of the Chern Simons description of the Quantum Hall fluid (for $\nu = 1/n$) given in [3]. The appropriate Chern Simons theory has as its gauge invariance the group of area preserving diffeomorphisms. In the linearized approximation it becomes a conventional abelian Chern Simons theory which efficiently describes the large distance physics including the charge and statistics of the quasiparticles. In a crude quantization one can see qualitatively, but not quantitatively, the origin of the quantization of the fluid density for the simplest filling fractions.

In order to correctly capture the granular structure of the fluid we upgraded the Chern Simons theory to a non-commutative gauge theory. The theory can also be thought of as a matrix theory of the elementary charges. The matrix theory is rich enough to describe fermions, bosons or anyons in a strong magnetic field. The non-commutative theory exactly reproduces the quantitative connection between filling fraction (level in the Chern Simons description) and statistics required by Laughlin's theory.

There are interesting predictions about the non-commutative Chern Simons theory that follow from the correspondence. An example is the phase transition between Quantum Hall fluid behavior and the Wigner crystal that occurs at low filling fraction. This suggests a phase transition in the non-commutative Chern Simons theory at large level. The

transition would be associated with the spontaneous breaking of a symmetry under area preserving diffeomorphisms of real x space. Similar phenomena have been observed fairly generically in non-commutative quantum field theories [15].

Finally in conclusion a speculation will be offered concerning the generalization to more general filling factors for electrons. For example consider the case $\nu = p/n$ with p and n relatively prime. One way to try to construct such a state is to first imagine p non-interacting layers of Quantum Hall fluid, each with filling fraction $1/n$. This can be represented in an obvious way by a matrix theory of block diagonal matrices where the number of blocks is p . We can think of the layers as branes separated by a large enough distance so that the electrons can't tunnel between them.

When the layers are adiabatically brought together so that the electrons are easily shared between them, the state must approach the fractional Quantum Hall state with $\nu = p/n$. Experience with D-branes suggests that the resulting theory should be a non-abelian version of the gauge theory. The natural guess is non-commutative Chern Simons $U(p)$ theory at level n .

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